

Current-induced spin torques in III-V ferromagnetic semiconductors

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We formulate a theory of current-induced spin torques in inhomogeneous III-V ferromagnetic semiconductors. The carrier spin-3/2 and large spin-orbit interaction, leading to spin non-conservation, introduce significant conceptual differences from spin torques in ferromagnetic metals. We determine the spin density in an electric field in the weak momentum scattering regime, demonstrating that the torque on the magnetization is intimately related to spin precession under the action of both the spin-orbit interaction and the exchange field characteristic of ferromagnetism. The spin polarization excited by the electric field is smaller than in ferromagnetic metals and, due to lack of angular momentum conservation, cannot be expressed in a simple closed vectorial form. Remarkably, scalar and spin-dependent scattering do not affect the result. We use our results to estimate the velocity of current-driven domain walls.

I. INTRODUCTION

Over a decade ago, Slonczewski¹ and Berger² predicted that an electrical current induces a torque on the magnetization of a ferromagnetic metal, and subsequent research has since identified distinct contributions called the reactive spin transfer torque and the dissipative spin transfer torque, sometimes referred to as adiabatic and non-adiabatic, respectively.³ Progress on the understanding of this effect in metals has been steady,^{4,5,6,7,8,9,10,11,12,13,14} and the field has been stimulated by applications in spintronics and nanotechnology, as a way to manipulate magnetization and thus information.

Current-induced spin torques arise from a small mismatch between the spin polarization of conduction electrons and the magnetization present throughout a material, and reflect the nonlocal nature of magnetization dynamics in inhomogeneous systems.¹⁵ They are the converse of processes such as giant magnetoresistance (GMR). The calculation of spin torques is equivalent to finding the conduction-electron spin density in an electric field, and in metals they can be easily expressed as the divergence of the spin current.¹⁴ Recently spin torque-related effects have been investigated in metal-based systems and nanomagnets, e.g., spin-torque driven ferromagnetic resonance,¹⁶ spin torques in nanomagnets¹⁷ and in continuously variable magnetizations,¹⁸ the s - d interaction in inhomogeneous ferromagnets,¹⁹ current-induced magnetization dynamics,²⁰ and domain wall motion.²¹

Despite these efforts, spin torques have been little studied in materials displaying some of the richest physics and technological promise, namely ferromagnetic semiconductors. The most intensely studied ferromagnetic semiconductors are Mn-doped III-V compounds, which fall into two classes: zincblende lattices, such as (Ga,Mn)As and wurtzite lattices, such as (Ga,Mn)N. The former have been much investigated^{22,23} and will be the sub-

ject of this work, while the latter are awaiting further experimental developments.²⁴ Recent experiments have succeeded in fabricating (Ga,Mn)As spin transfer torque devices,²⁵ while domain wall motion^{26,27} and resistance²⁸ have been addressed, and spin transfer physics was the subject of a recent review.²⁹

Ferromagnetic semiconductors differ profoundly from ferromagnetic metals. Being p -type, the carriers are holes described by an effective spin-3/2 and subject to a strong spin-orbit interaction. The hole spin is thus not conserved. The band Hamiltonian is the Luttinger Hamiltonian,³⁰ which, in the spherical approximation, is composed of a series of multipoles in spin space.³¹ The hole spin is a dipole, while the spin-orbit interaction is a quadrupole, and the precession of the spin in a quadrupole field is highly nontrivial.³² Magnetism is due to localized Mn moments, with the exchange interaction mediated by itinerant holes. In the mean-field approximation this interaction is taken into account as an effective field (a dipole) acting on the carriers and produces a splitting comparable in magnitude to the Fermi energy.²² The combined effect of this dipole exchange field and the quadrupole spin-orbit interaction is not a simple additive problem.³² These materials are often in the weak momentum scattering limit,²² i.e., $\varepsilon_F \tau / \hbar \gg 1$, where ε_F is the Fermi energy and τ is a characteristic scattering time. Due to the fast spin precession as a result of the spin-orbit interaction the relaxation time approximation is inappropriate to describe spin dynamics. The carrier spin is not conserved which introduces arbitrariness in the definition of the spin current.³³ Therefore, spin torques cannot be identified, as in metallic systems,¹⁴ with the divergence of the spin current. Finally electrically-induced spin densities³⁴ in bulk nonmagnetic zincblende semiconductors are forbidden by symmetry, raising the question of what form the spin density may take in such a system when the symmetry is lowered by the magnetization.

In this paper we present a general theory of current-induced spin torques in zincblende III-V ferromagnetic

semiconductors, which, to our knowledge, is the first of its kind. Our study is based on the Luttinger Hamiltonian³⁰ and it assumes $\varepsilon_F\tau/\hbar \gg 1$, a time-independent magnetization with small spatial gradients, zero temperature, no compensation and short-ranged impurity potentials (justified by the large carrier densities in ferromagnetic semiconductors and the short-range nature of the exchange interaction²²). We take the spin-orbit interaction to be the *dominant* term and treat the exchange field in first-order perturbation theory. This is a good approximation for the lower end of Mn concentrations ($\approx 2 - 5\%$). We neglect also quantum interference effects such as weak localization.

In recent years a number of microscopic theories of spin torques in ferromagnetic metals have been developed.^{11,12,13,14} A common strategy is to begin with a uniformly magnetized state and consider small perturbations around it. The method used in this work is slightly different, although the physics is the same and no difference is expected in the final results. We assume a magnetization that is a function of position and has small gradients, and we work to first order in the gradient of the magnetization. We would like to note that a recent study has been published which considers spin torques in metals starting from a spin continuity equation.³⁵

The outline of this paper is as follows. The next section contains a derivation of the kinetic equation that will be used in the remainder of the paper. We begin from the quantum Liouville equation and project it in momentum space, then introduce so-called Wigner coordinates and derive the equation satisfied by the Wigner distribution. In Section III this equation is solved in the presence of an electric field, and in section IV the spin density induced by the electric field is found, which gives the spin torque acting on the magnetization. The form and implications of the results are discussed, and their applicability to GaMnAs is demonstrated. Finally, the domain wall velocity as a result of the spin torque is estimated in Sec. V.

II. KINETIC EQUATION

The typical setup for a spin torque experiment consists of two slabs of ferromagnetic material with non-collinear magnetizations, separated by a tunnel barrier. Since the magnetizations of the two slabs are non-collinear there is a region near the interface over which the magnetization changes. To determine the continuum limit of this setup one can begin by visualizing a large number of slabs put together, with slight variations in the direction of the magnetization of each slab. Then one can imagine the interfaces between the slabs disappearing, leaving one large sample with an inhomogeneous magnetization, in such a way that the gradient of the magnetization varies little over distances comparable to the lattice spacing. The gradient expansion in the magnetization that follows from this procedure is valid as long as the length

scale on which the magnetization varies is much longer than the relevant length scales of the carriers, i.e., the Fermi wavelength and mean free path.

Spin torques appear when an electrical current flows through such a material. In a ferromagnetic semiconductor the magnetization is a result of the Mn ions which interact by means of the exchange coupling mediated by itinerant holes. The holes themselves have a spin polarization, and in equilibrium the hole spin polarization follows the magnetization. When an electrical current flows through the sample one can think for example of a hole which is taken from position \mathbf{r} , where its spin is parallel to the local magnetization at \mathbf{r} , and transporting it to $\mathbf{r} + \delta\mathbf{r}$. The magnetization at $\mathbf{r} + \delta\mathbf{r}$ is slightly different from the magnetization at \mathbf{r} , so a torque is exerted by the itinerant hole on the magnetization. What makes the situation in ferromagnetic semiconductors more difficult and more interesting is that while the hole is moving from \mathbf{r} to $\mathbf{r} + \delta\mathbf{r}$ it is subject to the strong spin-orbit interaction, which acts to randomize its spin. In order to determine the effect of spin-orbit interactions, which are wave vector-dependent, on the itinerant holes and ultimately on the inhomogeneous magnetization we need to study the kinetic equation, which takes into account both the momentum dependence and the position dependence.

In this section we will derive a kinetic equation suitable for describing inhomogeneous ferromagnetic semiconductors in an electric field. We consider the system to be described by a density operator $\hat{\rho}$, which obeys the quantum Liouville equation

$$\frac{d\hat{\rho}}{dt} + \frac{i}{\hbar} [\hat{H}, \hat{\rho}] = 0. \quad (1)$$

The total Hamiltonian \hat{H} contains contributions due to the band Hamiltonian \hat{H}_v , the scalar impurity potential, the exchange interaction between delocalized holes and localized Mn moments, and the electric field. These will be given below. The Liouville equation is projected onto a set of states $|u_{\mathbf{k}s}\rangle$ of definite wave vector \mathbf{k} and spin s , which are assumed to be Bloch functions and eigenstates of the Luttinger Hamiltonian \hat{H}_v . The matrix elements of $\hat{\rho}$ in this basis are $\rho_{ss'}(\mathbf{k}, \mathbf{k}') \equiv \rho(\mathbf{k}, \mathbf{k}')$ and are treated as matrices in spin space. \hat{H}_v is diagonal in \mathbf{k} and its matrix elements in this basis are $H_v \equiv H_v(\mathbf{k})$ ^{30,31}

$$H_v = \frac{\hbar^2}{2m} \left[\gamma_1 k^2 + \bar{\gamma} \left(\frac{5}{2} k^2 - 2(\mathbf{k} \cdot \mathbf{S})^2 \right) \right], \quad (2)$$

where \mathbf{S} is a vector of spin-3/2 matrices. The term proportional to the Luttinger parameter γ_1 gives the hole kinetic energy. For $k \neq 0$, the term proportional to $\bar{\gamma}$ separates the heavy hole (HH) and light hole (LH) states, i.e., it is the spin-orbit coupling that plays a central role in the present analysis. The Mn^{2+} ions give rise both to a net magnetic moment, through the hole-mediated exchange interaction, and to scattering, which has a scalar part and a spin-dependent part. These are contained in

the Hamiltonian H_{Mn} ,

$$H_{\text{Mn}}(\mathbf{r}) = \sum_I [\mathcal{U}(\mathbf{r} - \mathbf{R}_I) \mathbb{1} + \mathcal{V}(\mathbf{r} - \mathbf{R}_I) \mathbf{s}_I \cdot \mathbf{S}], \quad (3)$$

where the sum runs over the positions \mathbf{R}_I of the Mn^{2+} ions, with \mathbf{s}_I the Mn spin. We approximate the interactions represented by H_{Mn} as short-ranged, so that $\mathcal{U}(\mathbf{r} - \mathbf{R}_I) = U \delta(\mathbf{r} - \mathbf{R}_I)$ and $\mathcal{V}(\mathbf{r} - \mathbf{R}_I) = (J_{pd}/V) \delta(\mathbf{r} - \mathbf{R}_I)$, with J_{pd} the exchange constant between the localized Mn moments and the itinerant holes, and V the sample volume. The matrix elements of H_{Mn} in the basis $\{|u_{\mathbf{k}s}\rangle\}$ are decomposed into a part H_{pd} diagonal in \mathbf{k} , which gives the net magnetization \mathbf{M} ,

$$H_{\text{Mn}}^{\mathbf{k}=\mathbf{k}'} \equiv H_{pd} = \frac{N_{\text{Mn}} J_{pd}}{V} \mathbf{s} \cdot \mathbf{S} \equiv \mathbf{M} \cdot \mathbf{S}, \quad (4)$$

where N_{Mn} is the number of Mn^{2+} ions and we assume all Mn spins polarized setting $\mathbf{s}_I = \mathbf{s}$, and a part off-diagonal in \mathbf{k} , which causes spin-dependent scattering and will be given below. We will concentrate in this work on the case in which the exchange splitting is smaller

than the spin-orbit coupling at the Fermi energy, i.e., $|\mathbf{M}| \ll 2\gamma\hbar^2 k_F^2/m$, where k_F is the Fermi wave vector, and we work to first order in $m|\mathbf{M}|/(2\gamma\hbar^2 k_F^2)$.

In studying inhomogeneous magnetization one must account for real-space as well as wave-vector dependence, which is accomplished by defining a Wigner distribution. The Wigner function corresponding to the one-particle density matrix $\rho(\mathbf{k}, \mathbf{k}')$ is

$$f_{\mathbf{q}}(\mathbf{r}) = \int \frac{d^3 Q}{(2\pi)^3} e^{i\mathbf{Q} \cdot \mathbf{r}} \rho(\mathbf{q} + \mathbf{Q}/2, \mathbf{q} - \mathbf{Q}/2). \quad (5)$$

where $\mathbf{q} = (\mathbf{k} + \mathbf{k}')/2$ and $\mathbf{Q} = \mathbf{k} - \mathbf{k}'$. The next step is to derive an equation describing the time evolution of the Wigner distribution $f_{\mathbf{q}}(\mathbf{r})$. The kinetic equation for the Wigner function $f \equiv f_{\mathbf{q}}(\mathbf{r})$ is obtained by projecting the quantum Liouville equation onto the basis states $|u_{\mathbf{k}s}\rangle$, then making the transformation (5). The Hamiltonian \hat{H} is diagonal in wave vector. The first step, in which no approximations have been made, gives us the Liouville equation in terms of the so-called Wigner coordinates \mathbf{q} and \mathbf{Q}

$$\frac{\partial f_{\mathbf{q}}}{\partial t} + \frac{i}{\hbar} \int \frac{d^3 Q}{(2\pi)^3} e^{i\mathbf{Q} \cdot \mathbf{r}} (H_{\mathbf{q}+\mathbf{Q}} \rho_{\mathbf{q}-\mathbf{Q}} - \rho_{\mathbf{q}+\mathbf{Q}} H_{\mathbf{q}-\mathbf{Q}}) = -\frac{i}{\hbar} \sum_{\kappa} \int \frac{d^3 Q}{(2\pi)^3} e^{i\mathbf{Q} \cdot \mathbf{r}} (U_{\mathbf{q}+\kappa} \rho_{\kappa\mathbf{q}-} - \rho_{\mathbf{q}+\kappa} U_{\kappa\mathbf{q}-}). \quad (6)$$

To obtain a transparent kinetic equation it is necessary to expand $H_{\mathbf{q}\pm}$ around the wave vector \mathbf{q} , which requires some care. In this article we are working in a basis in which the functions depend on the wave vector \mathbf{q} , and we require a formulation of the kinetic equation that is manifestly gauge covariant. This means that the kinetic equation should not acquire additional terms if the basis functions are subjected to a \mathbf{q} -dependent rotation. The end result of this requirement is that ordinary \mathbf{q} -derivatives are replaced by covariant derivatives (\mathbf{r} -derivatives, denoted by ∇ , remain unchanged since the basis does not have position dependence.) The covariant \mathbf{q} -derivative is defined by $\frac{Df}{D\mathbf{q}} \equiv \frac{\partial f}{\partial \mathbf{q}} - i[\mathcal{R}, f]$, where the gauge connection matrix $\mathcal{R}_{ss'} = \langle u_{qs} | i \frac{\partial u_{qs'}}{\partial \mathbf{q}} \rangle$. All this implies that in our derivation we must replace the ordinary derivatives by covariant derivatives. Expanding $H_{\mathbf{q}\pm} \approx H_{\mathbf{q}} \pm \frac{\mathbf{Q}}{2} \cdot \frac{DH_{\mathbf{q}}}{D\mathbf{q}}$ equation (6) can be written as

$$\frac{\partial f_{\mathbf{q}}}{\partial t} + \frac{i}{\hbar} [H_{\mathbf{q}}, f_{\mathbf{q}}] + \frac{1}{2\hbar} \left\{ \frac{DH_{\mathbf{q}}}{D\mathbf{q}} \cdot \nabla f_{\mathbf{q}} \right\} = -\frac{i}{\hbar} \sum_{\kappa} \int \frac{d^3 Q}{(2\pi)^3} e^{i\mathbf{Q} \cdot \mathbf{r}} (U_{\mathbf{q}+\kappa} \rho_{\kappa\mathbf{q}-} - \rho_{\mathbf{q}+\kappa} U_{\kappa\mathbf{q}-}), \quad (7)$$

where $\{\}$ denotes an anti-symmetrized dot product, that is the symmetrized scalar product between vectors \mathbf{a} and \mathbf{b} is given by $\{\mathbf{a} \cdot \mathbf{b}\} \equiv \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$. The scattering term represented by the RHS is dealt with in the appendix.

In a constant and uniform electric field the total Hamiltonian contains an additional term containing the electromagnetic potential $V = e\mathbf{E} \cdot \hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the position operator. This term is diagonal in real space. Following the spirit of the derivation presented above, this term appears on the right side of the kinetic equation in the same way as U , and is expanded as follows

$$\frac{i}{\hbar} \int \frac{d^3 Q}{(2\pi)^3} e^{i\mathbf{Q} \cdot \mathbf{r}} \langle \mathbf{q}_+ | [V, \rho] | \mathbf{q}_- \rangle \approx -\frac{1}{\hbar} \nabla V \cdot \frac{Df_{\mathbf{q}}}{D\mathbf{q}}. \quad (8)$$

The spatial gradient of the external electrical potential is equal to the electric field $-\nabla V = \mathbf{E}$. In this work we will be studying the response of the system to linear order in the electric field.

When formulating a kinetic equation, which takes into account the variation of the Wigner function in real space as well as in momentum space, it is necessary to single out the length and wave-vector scales relevant to the problem under study. In the work at hand we consider carriers which are delocalized in real space and are described by Bloch states, for which the wave vector is a good quantum number. Nevertheless it must be borne in mind that the carrier occupies a finite range of real and

momentum space, denoted by $\Delta\mathbf{r}$ and $\Delta\mathbf{q}$ respectively, which are determined in such a way as to be consistent with the Heisenberg uncertainty principle. In the course of a scattering event in which a carrier with wave vector \mathbf{q} interacts with the potential of an impurity and its wave vector changes from \mathbf{q} to $\boldsymbol{\kappa}$, it is necessary as well as physical to assume that the wave-vector spread $\Delta\mathbf{q}$ as-

sociated with the carrier size is much smaller than the typical momentum transfer in scattering processes $\boldsymbol{\kappa} - \mathbf{q}$. Furthermore, it is assumed that the magnetization \mathbf{M} varies over length scales much larger than interatomic separations. With these assumptions, the kinetic equation in an electric field \mathbf{E} takes the form (in agreement with the form found by Carruthers and Zachariasen³⁷)

$$\frac{\partial f}{\partial t} + \frac{i}{\hbar} [H_v + H_{pd}, f] + \frac{1}{2\hbar} \left\{ \frac{D}{D\mathbf{q}} (H_v + H_{pd}) \cdot \nabla f \right\} - \frac{1}{2\hbar} \left\{ \nabla H_{pd} \cdot \frac{Df}{D\mathbf{q}} \right\} + \hat{J}(f) = \Sigma_E, \quad (9)$$

where $\Sigma_E = -e\mathbf{E} \cdot (Df/D\mathbf{q})$ is the covariant form of the usual source term due to \mathbf{E} . The term $\hat{J}(f)$ represents the scattering term, which is discussed in detail in the appendix. The scattering term takes into account the effect of the potential \mathcal{U} , which represents the part of the Hamiltonian H_{Mn} which is off-diagonal in wave vector. An explicit form for the scattering term will be given below when we discuss the solution of the kinetic equation in an electric field.

III. SOLUTION OF THE KINETIC EQUATION

The equilibrium distribution f_{eq} is the solution to Eq. (9) in the absence of external fields, $\Sigma_E = 0$. To leading order in $|U|^2$ this solution is $f_{\text{eq}} = f_0(H_v + H_{pd})$, with f_0 the Fermi-Dirac function. It is straightforward to check that this form of the Wigner function satisfies the kinetic equation (9) when the RHS is equal to zero. The form of this solution shows that in equilibrium the spin polarization of the holes follows the magnetization of the Mn, which is contained in the exchange part of the Hamiltonian H_{pd} .

Next, in the linear response regime, we search for a solution of the kinetic equation for nonzero Σ_E , which will yield the spin density induced by \mathbf{E} . Since the spin density induced by the electric field will be a function of position and will in general not be parallel to the local magnetization, this will immediately give the spin torque exerted by the conduction holes on the magnetization. The method we use to solve the kinetic equation is as follows. First, we divide every matrix \mathcal{M} in the problem into $\mathcal{M}^{\text{in}} + \mathcal{M}^{\text{out}}$, where \mathcal{M}^{in} has elements only within the HH and LH subspaces, while \mathcal{M}^{out} has matrix elements only between these subspaces. Schematically this can be summarized by

$$M = \begin{pmatrix} \text{in} & \text{out} \\ \text{out} & \text{in} \end{pmatrix}. \quad (10)$$

One compelling advantage of this decomposition is that commutators and anticommutators of matrices belonging to either the *in* or *out* sectors do not mix these sectors.

The following list covers all the possible combinations of commutators and anticommutators of matrices belonging to either the *in* or the *out* sectors

$$\begin{aligned} [in, out] &= out \\ [in, in] &= in \\ [out, out] &= out \\ \{in, out\} &= out \\ \{in, in\} &= in \\ \{out, out\} &= in. \end{aligned} \quad (11)$$

Another advantage of this decomposition is that it aids us in constructing a physical picture of spin torques and their relation to spin precession. The decomposition into an *in* and an *out* sector in effect singles out spin precession as a result of the spin-orbit interaction. The *in* sector of the density matrix represents spins that are stationary under the action of the spin-orbit interaction, or alternatively the fraction of the spins that are in eigenstates of H_v . The *out* sector on the other hand represents spins that precess under the action of the spin-orbit interaction. This decomposition determines which spin torques are due to the hole spin precession, which, unlike the precession of spin-1/2 electrons, cannot be attributed to an effective magnetic field.³² Being in the weak momentum scattering regime $\varepsilon_F \tau / \hbar \gg 1$, we do not consider scattering in the *out* sector or between the *in* and *out* sectors (it can be shown that both of these terms yield corrections linear in $|U|^2$). The Wigner function f has two parts, f^{in} in the *in* sector and f^{out} in the *out* sector, and the kinetic equation is broken down into two coupled equations for f^{in} and f^{out}

$$\frac{\partial f^{\text{in}}}{\partial t} + \frac{i}{\hbar} [H^{\text{in}}, f^{\text{in}}] + \hat{J}(f^{\text{in}}) = \Sigma_E^{\text{in}} + \Sigma_{\text{gr}}^{\text{in}} \quad (12a)$$

$$\frac{\partial f^{\text{out}}}{\partial t} + \frac{i}{\hbar} [H^{\text{in}}, f^{\text{out}}] = \Sigma_E^{\text{out}} + \Sigma_{\text{gr}}^{\text{out}}. \quad (12b)$$

There are two source terms in each equation, namely Σ_E^{in}

and Σ_{gr}^{in} in the *in* sector, and Σ_E^{out} and Σ_{gr}^{out} in the *out* sector. To obtain these source terms one needs to expand all quantities in the gradient of the magnetization and keep terms to zeroth and first order in this gradient. To zeroth order in the gradient of the magnetization the source terms are $\Sigma_E^{in/out} = (e\mathbf{E}/\hbar) \cdot (Df_{eq}/D\mathbf{q})^{in/out}$,

which are found simply by taking Σ_E defined above and substituting f_{eq} for the Wigner function. When the expansion is continued to the next order, the source terms linear in the gradient (gr) of the magnetization are

$$\Sigma_{gr}^{in} = \frac{1}{2\hbar} \left\{ \nabla H^{out} \cdot \frac{Df^{out}}{D\mathbf{q}} \right\} - \frac{1}{2\hbar} \left\{ \frac{DH^{out}}{D\mathbf{q}} \cdot \nabla f^{out} \right\} - \frac{1}{2\hbar} \left\{ \frac{DH^{in}}{D\mathbf{q}} \cdot \nabla f^{in} \right\} \quad (13a)$$

$$\Sigma_{gr}^{out} = \frac{1}{2\hbar} \left\{ \nabla H^{in} \cdot \frac{Df^{out}}{D\mathbf{q}} \right\} - \frac{1}{2\hbar} \left\{ \frac{DH^{in}}{D\mathbf{q}} \cdot \nabla f^{out} \right\} - \frac{i}{\hbar} [H^{out}, f^{out} + f^{in}]. \quad (13b)$$

To obtain Eq. (13) we have assumed a small spatial gradient $\nabla H = \nabla H_{pd}$ implying a small variation $\delta M \ll |\langle \mathbf{M} \rangle|$, and we worked, as stated, to first order in $m|\mathbf{M}|/(2\bar{\gamma}\hbar^2 k_F^2)$. After some simplification we obtain for the scattering term acting on f^{in}

$$\hat{J}(f^{in}) = \frac{f^{in} - \overline{f^{in}}}{\tau} + \frac{\overline{\Gamma_s^2}}{\tau} f^{in} - \frac{1}{\tau} \overline{\Gamma_s f^{in} \Gamma_s} + \frac{m^*}{\tau q^2 \hbar^2} \int \frac{d\Omega'}{4\pi} \left[(f^{in} - f^{in'}) (H_{pd} - H'_{pd}) - q \frac{\partial f^{in'}}{\partial q} (H_{pd} - H'_{pd}) \right], \quad (14)$$

where the bar is an average over directions in momentum space, $\tau^{-1} = N_{Mn}|U|^2 m^* q / (V\pi\hbar^3)$, $f \equiv f(q, \theta, \phi)$ and $f' \equiv f(q, \theta', \phi')$, θ and ϕ are the polar and azimuthal angles of \mathbf{q} (analogously for \mathbf{q}'), and m^* is the carrier effective mass, which is $m/(\gamma_1 - 2\bar{\gamma})$ in the HH subspace, and $m/(\gamma_1 + 2\bar{\gamma})$ in the LH subspace.

For simplicity and without loss of generality we choose $\mathbf{E} \parallel \hat{\mathbf{y}}$ and $\langle \mathbf{M} \rangle \parallel \hat{\mathbf{z}}$ so that $M_{x,y}(\mathbf{r}) \ll M_z(\mathbf{r})$. We solve Eqs. (12) as follows: the equation for f^{out} is first solved with Σ_E^{out} as the initial source, and the solution f_E^{out} thus obtained is substituted into Σ_{gr}^{out} and Σ_{gr}^{in} . The equation for f^{in} is solved in an analogous fashion. The solutions to the equations for f^{in} and f^{out} involve expressions of the form $e^{iH^{in}t/\hbar} \mathcal{M} e^{-iH^{in}t/\hbar}$, and f^{out} is easily found. This is because in the *out* sector the product $e^{iH^{in}t/\hbar} \mathcal{M}^{out} e^{-iH^{in}t/\hbar}$ contains only functions of time of the form $\sin \omega t$ and $\cos \omega t$, with $\omega = 2\hbar\bar{\gamma}q^2/m$ the energy difference between the HH and LH bands when the magnetization is zero. The steady-state solution for f^{out} therefore involves only a straightforward time integral of the kind customarily encountered in linear-response theories. The equation for f^{in} takes more effort due to the presence of the scattering term and we only summarize the method here (it is described in detail in a recent publication by two of us³⁸). The *in* sector represents the part of the Wigner function that is stationary under the action of H_v . Nevertheless, the full Hamiltonian is $H_v + H_{pd}$, and the commutator $[H_{pd}^{in}, f^{in}]$ is not zero. In a manner similar to the decomposition of f into f^{in} and f^{out} , f^{in} itself is split into a part that commutes with H_{pd}^{in} , and a part that does not. It can be shown³⁸ that the commuting part yields a correction to the Wigner function that is linear in τ while the non-commuting part gives a

correction that does not depend on τ . However, we find that all contributions to f average to zero over directions in momentum space except f_{gr}^{out} . This implies that all contributions from f^{in} average to zero over directions in momentum space. f_{gr}^{out} gives rise to a spin density \mathcal{S} that is independent of scattering. It is discussed in detail below.

IV. SPIN TORQUES

The only contribution to the spin density in an electric field comes from f_{gr}^{out} . The three components of the spin density \mathcal{S} that this correction to the Wigner function yields are

$$\mathcal{S}_x = \frac{eE_y m^{1/2}}{\varepsilon_F^{3/2}} \left(\eta_x \frac{\partial M_x}{\partial y} - \zeta_x \frac{\partial M_y}{\partial x} \right) \quad (15a)$$

$$\mathcal{S}_y = \frac{eE_y m^{1/2}}{\varepsilon_F^{3/2}} \left(\eta_y \frac{\partial M_y}{\partial y} - \zeta_y \frac{\partial M_x}{\partial x} \right) \quad (15b)$$

$$\mathcal{S}_z = \frac{eE_y m^{1/2}}{\varepsilon_F^{3/2}} \eta_z \frac{\partial M_z}{\partial y}. \quad (15c)$$

These equations are the central result of our work. The dimensionless quantities η_i and ζ_i , with $i = x, y, z$ are functions of the Luttinger parameters γ_1 and $\bar{\gamma}$. For GaMnAs we find (all $\times 10^{-4}$) $\eta_x = \eta_z = 3.66$, $\eta_y = 5.52$, $\zeta_x = 11.56$ and $\zeta_y = 6.16$. The steady-state spin density is not collinear with the magnetization, so there will be a torque on the magnetization giving a precession frequency of magnitude $J_{pd}|\mathcal{S}|$. Taking $p = 1.2 \times 10^{20} \text{ cm}^{-3}$,

$E_y = 100$ kV/m, and estimating the change in the magnetization as 20% over one lattice spacing, the time scale of this precession is 200 ns – less than in metals, but \mathbf{M} itself is also typically one order of magnitude smaller.

A. Discussion

The fact that the spin torque comes only from $f_{\text{gr}}^{\text{out}}$ implies that the steady-state spin density is due to precession under the action of both the spin-orbit interaction and the exchange field. The fraction of the spins that is conserved, which would yield a term $\propto \tau$, gives a contribution that averages to zero in momentum space. The quantities η_i and ζ_i decrease with increasing spin-orbit interaction (given by $\bar{\gamma}$), suggesting the spin-orbit interaction reduces the spin torque. This agrees with the finding that there is no electrically-induced spin density in the corresponding nonmagnetic systems,³⁴ i.e. in the limit of large spin-orbit interaction $\bar{\gamma}$. This limit is equivalent to restoring the spherical symmetry of the Luttinger Hamiltonian of Eq. (2), which in ferromagnetic semiconductors is broken by the magnetization.

An important difference from ferromagnetic metals is that, in Eq. (15), there is no contribution from scattering, either scalar or spin-dependent. This fact indicates that the dominant spin torque in ferromagnetic semiconductors in the weak momentum scattering limit is intrinsic. This observation agrees with the results of Jungwirth *et al.*,³⁹ who studied the anomalous Hall effect in ferromagnetic semiconductors in the regime $\varepsilon_F \tau / \hbar \gg 1$ and found similarly that the role of scattering is secondary. It is also related to the absence of electrically-induced spin polarization in bulk nonmagnetic zincblende materials. Generally, such a spin polarization is due to the fraction of spins that is conserved³⁸ and is linear in τ , but this spin polarization is forbidden by symmetry in zincblende lattices.³⁴ The magnetization breaks the cubic symmetry of the lattice and gives a steady-state spin density, but the term linear in τ still averages to zero. We come back to the comparison of our result with result found for ferromagnetic metals in the next section.

We find that an electric field $\mathbf{E} \parallel \hat{x}$ corresponds to the permutation $x \leftrightarrow y$ in Eq. (15). Yet for a given orientation of \mathbf{E} , unlike in ferromagnetic metals, in ferromagnetic semiconductors there is no symmetry between the different components of the spin density for the following reason. In metals spin is conserved and spin torques can be derived phenomenologically directly from the Landau-Lifshitz-Gilbert equation (the so-called *book-keeping* argument^{8,11}). One assumes an itinerant spin passes a localized moment at \mathbf{r} , lines up with it, then moves on to another moment at $\mathbf{r} + \delta\mathbf{r}$ and exerts a torque on this moment. This relates $\mathbf{M}(\mathbf{r} + \delta\mathbf{r})$ to $\mathbf{M}(\mathbf{r})$ and gives a simple vector-product form for $\mathbf{S}(\mathbf{r})$.^{11,12,13,14} In ferromagnetic semiconductors the spin-orbit interaction acts to randomize the itinerant spin moving between \mathbf{r} and $\mathbf{r} + \delta\mathbf{r}$, and there is no simple relationship between

$\mathbf{M}(\mathbf{r} + \delta\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$. Such a book-keeping argument is thus not valid and there is no symmetry in the final expression for the spin density.

We would like to comment on one last aspect of the relationship between the hole spin polarization and the magnetization in ferromagnetic semiconductors. The calculation presented in this work relies on a mean-field description of the magnetization and hole spin polarization. In this picture the itinerant holes are subject to an average magnetic field due to the Mn^{2+} ions, and the Mn^{2+} ions in turn are subject to the itinerant hole spin polarization, which can also be regarded as an average magnetic field.²² Since it is assumed that the spin-orbit interaction has spherical symmetry, there is no easy axis for the magnetization in the absence of an electric field. However, once the electric field is applied it is natural to ask whether the direction of the electric field provides an easy axis for the magnetization, in other words whether the magnetization in the direction of the electric field increases. We find that this indeed is true, but the increase in the magnetization is second order in the ratio H_{pd}/ε_F and is not significant.

In ferromagnetic metals, in which spin-orbit coupling is negligible, angular momentum is conserved. As a result spin torques in these materials can be encapsulated into a set of simple, compact, rotationally-invariant vectorial expressions. In ferromagnetic semiconductors, in which spin-orbit interactions are usually the dominant energy scale, angular momentum is not conserved and the final expressions for the spin torques cannot be expected to have rotational invariance. In principle spin-orbit interactions, which couple the spin and the lattice, should give magnetic anisotropy and anisotropic spin torques as well. The anisotropy in our result for the spin density is thus a direct result of the intrinsic spin-orbit interactions.

B. Parameters and applicability for GaMnAs

We shall assume a doping density $n_{\text{Mn}} = p = 1.2 \times 10^{20} \text{ cm}^{-3}$, corresponding to $x = 2.2\%$, $J_{pd} = 54$ meV nm³ as discussed in Ref. [22] and the lattice constant $a = 5.6533 \text{ \AA}$. The Fermi energy is found as

$$\left(\frac{2m_h \varepsilon_F}{\hbar^2}\right)^{3/2} + \left(\frac{2m_l \varepsilon_F}{\hbar^2}\right)^{3/2} = 3\pi^2 n \quad (16)$$

$$\varepsilon_F = 1.633 \frac{\hbar^2}{2m_0} (3\pi^2 n)^{2/3} = 2.1 \times 10^{-20} \text{ J}$$

and the heavy and light hole Fermi wave vectors are $k_h = 1.43 \times 10^9 \text{ m}^{-1}$ and $k_l = 0.55 \times 10^9 \text{ m}^{-1}$. The heavy hole and light hole masses are $m_h = 0.538 \times 10^{-30} \text{ kg}$ and $m_l = 0.076 \times 10^{-30} \text{ kg}$. These numbers also give the magnitude of the effective field $|H_{pd}| = n_{\text{Mn}} J_{pd} < S_{\text{Mn}} > = 2.52 \times 10^{-21} \text{ J}$, meaning that the ratio $|H_{pd}|/\varepsilon_F = 0.12$, so it is safe to do perturbation theory.

We also want to work out $\varepsilon_F \tau_p / \hbar$. The Fermi energy is $2.1 \times 10^{-20} \text{ J}$, which means $\varepsilon_F \tau_p / \hbar > 1$ for any momentum scattering time $\tau_p \geq 5 \times 10^{-15} \text{ s}$. For example

for $\varepsilon_F \tau_p / \hbar \approx 10$ we require $\tau_p = 5 \times 10^{-14}$ s, corresponding to a light-hole mobility of approximately $1000 \text{ cm}^2/\text{Vs}$ and a heavy-hole mobility of approximately $200 \text{ cm}^2/\text{Vs}$. Thus the theory is on very firm ground even for extremely low mobilities.

V. DOMAIN-WALL MOTION

As an application of the central result in Eq. (15) we calculate the spin torque on a domain wall, and the resulting domain-wall velocity. We choose the current and variation of magnetization in the y -direction. Furthermore, we use $\eta_x = \eta_z$ such that Eq. (15) reduces to

$$\mathcal{S} = \frac{e E_y m^{1/2} \eta_x}{\varepsilon_F^{3/2}} \frac{\partial}{\partial y} \left[\mathbf{M} + \left(\frac{\eta_y}{\eta_x} - 1 \right) M_y \hat{\mathbf{y}} \right], \quad (17)$$

with $\hat{\mathbf{y}}$ a unit vector in the y -direction. The spin-transfer torque that acts on the magnetization is given by

$$\left. \frac{\partial \mathbf{M}}{\partial t} \right|_{\text{current}} = -\frac{J_{\text{pd}}}{\hbar^2} \mathbf{M} \times \mathcal{S}. \quad (18)$$

Using Eq. (15) we rewrite this as an equation for a unit vector $\boldsymbol{\Omega}$ in the direction of magnetization, i.e., $\mathbf{M} = n_{\text{Mn}} J_{\text{pd}} S_{\text{Mn}} \boldsymbol{\Omega}$, with $S_{\text{Mn}} = 5/2$ the spin of one Mn atom. We find that

$$\left. \frac{\partial \boldsymbol{\Omega}}{\partial t} \right|_{\text{current}} = -v \boldsymbol{\Omega} \times \frac{\partial}{\partial y} \left[\boldsymbol{\Omega} + \left(\frac{\eta_y}{\eta_x} - 1 \right) \Omega_y \hat{\mathbf{y}} \right], \quad (19)$$

with the velocity v given by

$$v = \frac{n_{\text{Mn}} e E m^{1/2} \eta_x J_{\text{pd}}^2 S_{\text{Mn}}}{\hbar^2 \varepsilon_F^{3/2}}. \quad (20)$$

The result for the current-induced torques in Eq. (19) has the form of an anisotropic dissipative spin transfer torque.¹¹ The reactive spin transfer torque contribution is equal to zero. These results are understood by noting that we have considered strong spin-orbit interactions, and that have done perturbation theory in the magnetization.

It is common to define a dissipative coefficient^{11,13} β such that $v \sim \beta j$, with j the current density. Because our result for the spin transfer torque is independent of τ , and because $j \sim \tau$, we would find that $\beta \sim 1/\tau$, i.e., resistivity-like. This is somewhat surprising as recent studies⁴¹ indicate that the Gilbert damping constant α_G , which is believed to be similar though not exactly equal to β , predominantly has intra-band contributions that are conductivity-like. However, a direct comparison is not possible because in the present paper we perform an expansion in the magnitude of the magnetization whereas Ref. [41] calculates α_G by determining the transverse response function.

The velocity v divided by the Gilbert damping constant provides an estimate for the domain wall velocity \dot{X} ,⁸ so that

$$\dot{X} \sim \frac{v}{\alpha_G}. \quad (21)$$

Although Sinova *et al.* do not explicitly consider the regime of parameters quoted in the previous section, their calculations⁴⁰ (see also Ref. [41]) suggest that the Gilbert damping is very small $\alpha_G \sim 0.001$ in this regime. Using this result we find that $\dot{X} \sim 1$ m/s, in agreement with experimental results for the domain-wall velocity.²⁷

To investigate more quantitatively the effect of the anisotropy in the spin transfer torque, determined by the ratio η_y/η_x , we consider specific model for a magnetic domain wall. We consider a thin film, in which there is a constant hard-axis anisotropy K_\perp perpendicular to the film and an easy-axis anisotropy K_E . Within the model for a rigid domain wall proposed by Tatara and Kohno⁴² (see also Ref. [43]), the domain wall is described by two collective coordinates: the position $X(t)$ and the chirality $\phi_0(t)$. The chirality is the angle with which the magnetic moment in the center of the domain wall tilts out of the easy plane. Using the results from Ref. [42] and Ref. [43] we find the equations of motion for the domain-wall collective coordinates. They are given by

$$\frac{\dot{X}}{\lambda} - \alpha_G \dot{\phi}_0 = \frac{K_\perp}{\hbar} \sin 2\phi_0 \quad (22a)$$

$$\dot{\phi}_0 + \alpha_G \frac{\dot{X}}{\lambda} = \frac{v}{\lambda} (1 + \delta \cos^2 \phi_0), \quad (22b)$$

where $\lambda = \sqrt{J/K_E}$ is the width of the domain wall. Note that δ goes to zero for $\eta_x \rightarrow \eta_y$. Note that, in addition usual dissipative spin transfer torque contribution to these equations discussed in earlier work,^{42,43} we find a chirality-dependent anisotropic contribution proportional to δ .

The above equation can be solved analytically. From this we obtain the average drift velocity as a function of the applied electric field, as shown in Fig. 1. From this figure, we observe a Walker-breakdown-like behavior,⁴⁴ i.e., the domain wall velocity reaches a maximum and then becomes smaller. Physically, the breakdown is due to the transition of rigid motion to precessional motion of the domain wall and is well-known from field-driven domain-wall motion. Our results are understood from the fact that dissipative spin-transfer torque enters the equations of motion for the domain wall in the same way as an external magnetic field. Note that the anisotropy δ alters the result for the domain-wall velocity somewhat with respect to the isotropic ($\delta = 0$) situation, but plays no qualitatively important role.

As a final remark, we note that in the calculations presented here we have neglected the effects of finite temperature⁴³ and pinning of the domain wall. This, in addition to the fact that the experiments of Yamanouchi *et al.*²⁷ are in a different regime of doping than consid-

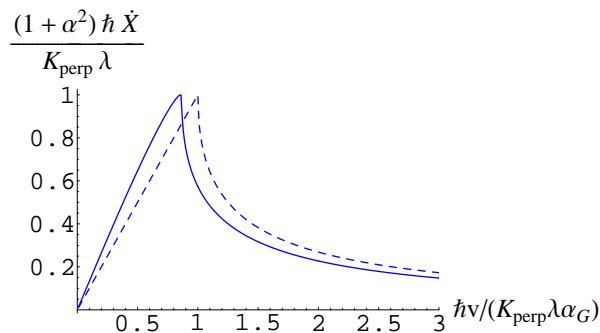


FIG. 1: Average drift velocity as a function of the applied electric field and for $\delta = 1/3$ (solid line) and $\delta = 0$ (dashed line).

ered here, makes a direct quantitative comparison not possible.

VI. CONCLUSIONS AND ACKNOWLEDGEMENTS

In conclusion, we have established a microscopic theory of spin transfer in III-V ferromagnetic semiconductors for

the case of strong spin-orbit coupling. We have applied our results to the case of current-driven domain wall motion and have estimated the resulting domain-wall velocities. We find domain-wall velocities that are of the same order of magnitude as experiments, although the available experimental results²⁷ are in a different regime of parameters than considered in this paper. Therefore, a more quantitative comparison between theory and experiment is at present not feasible.

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APPENDIX A: SCATTERING TERM

The scattering term $\hat{J}(f)$ is

$$\hat{J}(f) \approx \frac{N_{\text{Mn}}}{\hbar^2} \lim_{\eta \rightarrow 0} \int_0^\infty dt' e^{-\eta t'} \left[\hat{U}, e^{-i\hat{H}t'} [\hat{U}, \hat{f}] e^{i\hat{H}t'} \right]_{\mathbf{q}\mathbf{q}}, \quad (\text{A1})$$

with η a regularization factor and the impurity average $\langle H_{\text{Mn}}^{\mathbf{q}\mathbf{q}'} H_{\text{Mn}}^{\mathbf{q}'\mathbf{q}} \rangle_{\mathbf{q} \neq \mathbf{q}'} = N_{\text{Mn}} |U|^2 (\mathbb{1} + \Gamma_s)$, where N_{Mn} is the number of Mn impurities and $\Gamma_s = J_{pd}^2 (\mathbf{s} \cdot \mathbf{S})^2 / (|U|^2 V^2)$. The derivation of this general form of the scattering term is discussed in a recent paper³² and the notation will be explained in detail below. In terms of the Wigner distribution the scattering term can be expressed as

$$\begin{aligned} \hat{J}(f_{\mathbf{q}}) \approx & \frac{n_{\text{Mn}}}{\hbar^2} \int \frac{d^3\kappa}{(2\pi)^3} \int_0^\infty dt' e^{-\eta t'} U_{\mathbf{q}\kappa} e^{-iH_{\kappa}t'} (U_{\kappa\mathbf{q}} f_{\mathbf{q}} - f_{\kappa} U_{\kappa\mathbf{q}}) e^{iH_{\mathbf{q}}t'} \\ & - \frac{n_{\text{Mn}}}{\hbar^2} \int \frac{d^3\kappa}{(2\pi)^3} \int_0^\infty dt' e^{-\eta t'} e^{-iH_{\mathbf{q}}t'} (U_{\mathbf{q}\kappa} f_{\kappa} - f_{\mathbf{q}} U_{\mathbf{q}\kappa}) e^{iH_{\kappa}t'} U_{\kappa\mathbf{q}}. \end{aligned} \quad (\text{A2})$$

We must note that, in the approximation we are using, the scattering term acts only on f^{in} , which brings about some simplifications. These become apparent if we look

at the explicit form of this term and note that, because it involves only f^{in} , this term commutes with the time evolution operators.

$$\begin{aligned} \hat{J}(f^{in}) = & \frac{n_{\text{Mn}}}{\hbar^2} \int \frac{d^3\kappa}{(2\pi)^3} \int_0^\infty dt' e^{-\eta t'} U (e^{-iH_{\kappa}t'} U e^{iH_{\mathbf{q}}t'} f_{\mathbf{q}}^{in} - f_{\kappa}^{in} e^{-iH_{\kappa}t'} U e^{iH_{\mathbf{q}}t'}) \\ & - \frac{n_{\text{Mn}}}{\hbar^2} \int \frac{d^3\kappa}{(2\pi)^3} \int_0^\infty dt' e^{-\eta t'} (e^{-iH_{\mathbf{q}}t'} U e^{iH_{\kappa}t'} f_{\kappa}^{in} - f_{\mathbf{q}}^{in} e^{-iH_{\mathbf{q}}t'} U e^{iH_{\kappa}t'}) U. \end{aligned} \quad (\text{A3})$$

In the approximation used in this paper, the Hamiltonian entering the scattering term is the projected 2×2 Hamiltonian for each subspace. The scattering potential has two parts, one a scalar and one which is spin-dependent. Taking into account also the exchange splitting of the bands, there are three contributions to the scattering term: scalar potential + kinetic energy ($\equiv \hat{J}_0$), spin-dependent potential + kinetic energy ($\equiv \hat{J}_m$), scalar potential + exchange energy ($\equiv \hat{J}_s$). The former two sum up to

$$\begin{aligned} \hat{J}_0(f^{in}) + \hat{J}_m(f^{in}) \approx & \frac{2\pi n_{\text{Mn}}}{\hbar} \int \frac{d^3\kappa}{(2\pi)^3} \left(\frac{1}{2} \{U^2, f_{\mathbf{q}}^{in}\} - U f_{\kappa}^{in} U \right) \delta \left(\frac{\hbar^2 \kappa^2}{2m^*} - \frac{\hbar^2 q^2}{2m^*} \right) \\ = & \frac{n_{\text{Mn}} m^* q}{4\pi^2 \hbar^3} \int d\Omega' \left(\frac{1}{2} \{U^2, f_{\mathbf{q}}^{in}\} - U f_{\mathbf{q}}^{in'} U \right) = \frac{n_{\text{Mn}} m^* q}{\pi \hbar^3} \left(\frac{1}{2} \{\overline{U^2}, f_{\mathbf{q}}^{in}\} - \overline{U f_{\mathbf{q}}^{in} U} \right) \\ = & \frac{1}{\tau} \left(\frac{1}{2} \{\overline{\Gamma^2}, f_{\mathbf{q}}^{in}\} - \overline{\Gamma f_{\mathbf{q}}^{in} \Gamma} \right) \\ \frac{1}{\tau} = & \frac{n_{\text{Mn}} |U|^2 m^* q}{\pi \hbar^3} \quad \text{and} \quad U = |U| \Gamma \\ \overline{f_{\mathbf{q}}^{in}} = & \frac{1}{4\pi} \int d\Omega' f_{\mathbf{q}}^{in'}. \end{aligned} \quad (\text{A4})$$

We have used the notation $f_{\mathbf{q}}^{in'} \equiv f^{in}(q, \Omega')$. $|U|^2$ is a scalar and Γ is a dimensionless matrix, which is written as $\Gamma = \mathbb{1} + \Gamma_s$, with $\Gamma_s = 1/2 \Gamma_s \cdot \boldsymbol{\sigma}$. Notice that Γ_s has angular dependence because we are in the basis of eigenstates

of the Luttinger Hamiltonian. Thus these two contributions to the scattering term can be rewritten as

$$\begin{aligned} \hat{J}_0(f) + \hat{J}_m(f) &= \frac{1}{\tau} \left[\frac{1}{2} \{ \overline{(\mathbb{1} + \Gamma_s)^2}, f \} - \overline{(\mathbb{1} + \Gamma_s) f (\mathbb{1} + \Gamma_s)} \right] = \frac{1}{\tau} \left[f + \overline{\left\{ \left(\Gamma_s + \frac{\Gamma_s^2}{2} \right), f \right\}} - (\bar{f} + \overline{\{ \Gamma_s, f \}} + \overline{\Gamma_s f \Gamma_s}) \right] \\ &= \frac{f - \bar{f}}{\tau} + \frac{1}{\tau} \overline{\left\{ \Gamma_s + \frac{\Gamma_s^2}{2}, f \right\}} - \frac{1}{\tau} \overline{\{ \Gamma_s, f \}} - \frac{1}{\tau} \overline{\Gamma_s f \Gamma_s}. \end{aligned} \quad (\text{A5})$$

We think of τ as a characteristic scattering time. The explicit form of the potential, determined previously, is

$$U^2 = N_i \mathcal{U}^2 \left[1 + 2\alpha (\mathbf{s} \cdot \mathbf{S}) + \alpha^2 (\mathbf{s} \cdot \mathbf{S})^2 \right], \quad (\text{A6})$$

where $\alpha = \mathcal{V}/\mathcal{U}$. Everything must be averaged over the impurity configuration as well as directions in momentum space, and then it needs to be transformed into the eigenstate basis and projected onto LH and HH. When we do that, the term linear in α above contains only S_z , which, when projected onto LH and HH gives something that averages to zero over angles. Moreover, the configuration average of Γ_s^2 gives something which, when restricted to the HH and LH subspaces, is proportional to the identity matrix, so contributes only the scalar part of the scattering term.

$$\hat{J}_0(f) + \hat{J}_m(f) = \frac{f - \bar{f}}{\tau} + \frac{\overline{\Gamma_s^2}}{\tau} f - \frac{1}{\tau} \overline{\{ \Gamma_s, f \}} - \frac{1}{\tau} \overline{\Gamma_s f \Gamma_s}. \quad (\text{A7})$$

We separate the action of \hat{J}_m on the scalar and spin-dependent parts n , and S , of the Wigner distribution $f = n\mathbb{1} + S$. First on n , which is written as $n = \bar{n} + \nu$, where ν is the anisotropic part

$$\hat{J}_0(n) + \hat{J}_m(n) = (1 + \overline{\Gamma_s^2}) \frac{\nu}{\tau} - \frac{2}{\tau} \overline{\Gamma_s \nu} - \frac{1}{\tau} \overline{\Gamma_s^2 \nu} \quad (\text{A8})$$

Averaged over impurities Γ_s^2 gives

$$\Gamma_s^2 = N_i |\mathcal{V}|^2 [s_x^2 (S_x^2 + S_y^2) + s_z^2 S_z^2] = \gamma_{h,l}^2 \mathbb{1}, \quad (\text{A9})$$

the latter identity being valid because the matrix elements of the S_i^2 restricted to the HH and LH subspaces are proportional to the identity matrix. We also need to average $\Gamma_s \sigma_i \Gamma_s$, for which we note that $\sigma_i \sigma_j \sigma_i = -\sigma_j$

for $i \neq j$. Averaged over impurities

$$\begin{aligned} \Gamma_s \sigma_x \Gamma_s &= \frac{1}{4} (\Gamma_x^2 - \Gamma_y^2 - \Gamma_z^2) \sigma_x \\ \Gamma_s \sigma_y \Gamma_s &= \frac{1}{4} (\Gamma_y^2 - \Gamma_x^2 - \Gamma_z^2) \sigma_y \\ \Gamma_s \sigma_z \Gamma_s &= \frac{1}{4} (\Gamma_z^2 - \Gamma_x^2 - \Gamma_y^2) \sigma_z. \end{aligned} \quad (\text{A10})$$

This tells us that in the term $\overline{\Gamma_s S \Gamma_s}$ only the average of S , which we shall call \bar{S} , survives. Then, writing $S = \bar{S} + \Xi$

$$\hat{J}_0(S) + \hat{J}_m(S) = \frac{\Xi}{\tau} + \frac{\overline{\Gamma_s^2}}{\tau} (\bar{S} + \Xi) - \frac{1}{\tau} \overline{\{ \Gamma_s, \Xi \}} - \frac{1}{\tau} \overline{\Gamma_s \bar{S} \Gamma_s}. \quad (\text{A11})$$

Looking at Eq. (A8) and (A11) we see that if we ignore the term linear in Γ_s in each of them then they do not mix the scalar and spin distributions. We shall work for now in this approximation, which is justified because the terms omitted are higher order in the disorder potential. Then we can write

$$\begin{aligned} \hat{J}_0(n) + \hat{J}_m(n) &= \frac{\nu}{\tau_\gamma} - \frac{1}{\tau} \overline{\Gamma_s^2 \nu} \\ \frac{1}{\tau_\gamma} &= \frac{1 + \overline{\Gamma_s^2}}{\tau} \end{aligned} \quad (\text{A12})$$

$$\hat{J}_0(S) + \hat{J}_m(S) = \frac{\Xi}{\tau} + \frac{\overline{\Gamma_s^2}}{\tau} (\bar{S} + \Xi) - \frac{1}{\tau} \overline{\Gamma_s \bar{S} \Gamma_s}.$$

The contribution to the scattering term due to the exchange splitting of the bands is

$$\hat{J}_s(f) = \frac{m^*}{\tau q^2 \hbar^2} (f - \bar{f}) H_{pd} - \frac{m^*}{\tau q \hbar^2} \frac{\partial \bar{f}}{\partial q} H_{pd} + \frac{m^*}{\tau q^2 \hbar^2} \int \frac{d\Omega'}{4\pi} (f' + q \frac{\partial f'}{\partial q}) H'_{pd}. \quad (\text{A13})$$